BOUNDING THE INTEGRAL OF POWERED i-TH MEAN CURVATURES

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ABSTRACT. We get estimates for the integrals of powered i-th mean curvatures, $1 \leq i \leq n-1$, of compact and convex hypersurfaces, in terms of the quermaßintegrals of the corresponding C_+^2 convex bodies. These bounds will be obtained as consequences of a most general result for functions defined on a general probability space. From this result, similar estimates for the integrals of any convex transformation of the elementary symmetric functions of the radii of curvature of C_+^2 convex bodies will be also proved, both, in terms of the quermaßintegrals, and of the roots of their Steiner polynomials. Finally, the radial function is considered, and estimates of the corresponding integrals are obtained in terms of the dual quermaßintegrals.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{K}_0^n$ be the set of all convex bodies, i.e., compact convex sets with non-empty interior, in the n-dimensional Euclidean space $\mathbb{R}^n$, containing the origin. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively. The boundary of a set $M \subset \mathbb{R}^n$ is denoted by $\mathrm{bd}\, M$ and its convex hull by $\mathrm{conv}\, M$. We write B_2^n to denote the Euclidean unit ball centered at the origin, $B_2^n(r)$ for the ball (centered at 0) with radius $r > 0$, and $\mathbb{S}^{n-1} = \left\{ x \in \mathbb{R}^n : \|x\| = 1 \right\}$ for the unit sphere in $\mathbb{R}^n$. Let e_i denote the i-th canonical unit vector.

A convex body $K \in \mathcal{K}_0^n$ is said to be of class C^2 if its boundary hypersurface $\mathrm{bd}\, K$ is a regular submanifold of $\mathbb{R}^n$, in the sense of differential geometry, which is twice continuously differentiable. Moreover, we say that K is of class C_+^2 if K is of class C^2 and the Gauss map $\nu_K : \mathrm{bd}\, K \longrightarrow \mathbb{S}^{n-1}$, mapping a boundary point $x \in \mathrm{bd}\, K$ to the (unique) normal vector of K at x, is a diffeomorphism.

If M is a (proper) Borel subset of $\mathbb{R}^n$, we will use the same notation $|\cdot|$ for its volume (i.e., its Lebesgue measure) and its surface area, writing $|M|$ and $|\operatorname{bd} M|$, respectively. Moreover, by $\mathcal{H}^k$, $0 \le k \le n$, we denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and thus, if M is a subset of a k-plane or a k-dimensional sphere, then $\mathcal{H}^k(M)$ coincides, respectively, with the k-dimensional Lebesgue measure of M in $\mathbb{R}^k$ or with the k-dimensional spherical Lebesgue measure in $\mathbb{S}^k$. Moreover, the *outer radius* and the *inner radius* of $K \in \mathcal{K}_0^n$ are defined as the quantities

$$\bar{\mathrm{R}}(K) = \min\{R > 0 : K \subseteq RB_2^n\} = \max\{\|x\| : x \in \operatorname{bd} K\},$$

$$\bar{\mathrm{r}}(K) = \max\{r \ge 0 : rB_2^n \subseteq K\} = \min\{\|x\| : x \in \operatorname{bd} K\}.$$

Clearly, the value $\bar{\mathrm{R}}(K) - \bar{\mathrm{r}}(K)$ is not translation invariant, but since K is compact, there exists a point c_K such that

$$\bar{\mathrm{R}}(K - c_K) - \bar{\mathrm{r}}(K - c_K) = \min\{\bar{\mathrm{R}}(K - t) - \bar{\mathrm{r}}(K - t) : t \in K\}.$$

We observe that c_K is the so-called *center of the minimal annulus*, i.e., the uniquely determined annulus (closed set consisting of all points between two concentric balls) with minimal difference of radii containing $\operatorname{bd} K$.

If K is 0-symmetric, i.e., if $K = -K$, then $\bar{\mathrm{R}}(K)$ and $\bar{\mathrm{r}}(K)$ coincide with the classical *circumradius* and *inradius* of K, namely,

$$\mathrm{R}(K) = \min\{R : \exists\, x \in \mathbb{R}^n \text{ with } K \subseteq x + RB_2^n\},$$

$$\mathrm{r}(K) = \max\{R : \exists\, x \in \mathbb{R}^n \text{ with } x + RB_2^n \subseteq K\},$$

respectively. In general, we will say that a convex body K is *centered* if $\mathrm{R}(K) = \bar{\mathrm{R}}(K)$ and $\mathrm{r}(K) = \bar{\mathrm{r}}(K)$.

Given a compact (oriented) hypersurface $M \subset \mathbb{R}^n$ with smooth boundary and mean curvature H positive everywhere, Willmore's inequality says that

$$(1.1) \qquad \int_{\operatorname{bd} M} H^{n-1}\, \mathrm{d}\mathcal{H}^{n-1} \ge n|B_2^n|.$$

This was proved by Willmore [19] in the 3-dimensional case and by Chen in an arbitrary dimension, see [3, 4]. Moreover, Ros [15] proved that

$$(1.2) \qquad \int_{\operatorname{bd} M} \frac{1}{H}\, \mathrm{d}\mathcal{H}^{n-1} \ge n|M|.$$

Besides the major importance that these results have by themselves, they are specially interesting because they imply isoperimetric inequalities (see e.g. [14]). Further information and related results on this topic can be found in, e.g., [1, 11] and the references given there.

In the particular case of a convex body $K \in \mathcal{K}_0^n$ of class C_+^2, i.e., when the hypersurface $\operatorname{bd} K$ is convex, (1.1) can be trivially obtained from the relation $H^{n-1} \ge \kappa$, where κ denotes the Gauss-Kronecker curvature. These inequalities have been considered in a more general setting than convex hypersurfaces (see, e.g. [13]). However, a natural problem would be to get improvements of (1.1) and (1.2) when we remain in the convex case. At this respect, in [7] the planar case is considered, and lower bounds for the

integrals of powers of the curvature of a smooth bounded planar convex curve are shown.

One aim of the paper is to get estimates for the integrals of powered i-th mean curvatures H_i, $i = 1, \ldots, n-1$, of compact and convex hypersurfaces, which include, in particular, the mean curvature ($i = 1$) and the Gauss-Kronecker curvature ($i = n-1$). The bounds will be given in terms of the radii $\bar{\mathrm{r}}, \bar{\mathrm{R}}$ and the quermaßintegrals W_j of the corresponding convex bodies, which are special geometric measures associated to the set and including, in particular, the volume and the surface area (see Section 3 for the precise definition and properties).

Theorem 1.1. *Let* $K \in \mathcal{K}_0^n$ *be of class* C_+^2. *Then, for any* $\alpha \geq 0$ *and all* $0 \leq i \leq n-1$,

$$
\int_{\mathrm{bd}\,K} H_i^{\alpha+1} \mathrm{d}\mathcal{H}^{n-1} \geq \frac{n^{\alpha+1}}{2} \left[\frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(|\mathrm{bd}\,K| + \frac{\xi_i}{\mathrm{W}_{i+1}} \right)^\alpha} + \frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(|\mathrm{bd}\,K| - \frac{\xi_i}{\mathrm{W}_{i+1}} \right)^\alpha} \right],
$$

$$
\int_{\mathrm{bd}\,K} \frac{1}{H_i^\alpha} \mathrm{d}\mathcal{H}^{n-1} \geq \frac{1}{2n^\alpha} \left[\frac{|\mathrm{bd}\,K|^{\alpha+1}}{\left(\mathrm{W}_{i+1} + \frac{\xi_i}{|\mathrm{bd}\,K|} \right)^\alpha} + \frac{|\mathrm{bd}\,K|^{\alpha+1}}{\left(\mathrm{W}_{i+1} - \frac{\xi_i}{|\mathrm{bd}\,K|} \right)^\alpha} \right],
$$

where

$$
\xi_i = \begin{cases} \dfrac{|\mathrm{bd}\,K|\mathrm{W}_i - n|K|\mathrm{W}_{i+1}}{\bar{\mathrm{R}}(K - c_K) - \bar{\mathrm{r}}(K - c_K)} & \text{if } K \neq B_2^n(r) \text{ for any } r > 0, \\ 0 & \text{if } K = B_2^n(r) \text{ for some } r > 0. \end{cases}
$$

Equality holds in both inequalities if $K = B_2^n$ *(up to dilations).*

In particular, lower bounds for the integrals of the square/$(n-1)$-power and the inverse of the classical mean and Gauss-Kronecker curvatures can be obtained:

Corollary 1.1. *Let* $K \in \mathcal{K}_0^n$ *be of class* C_+^2. *Then,*

$$
\int_{\mathrm{bd}\,K} H^{n-1} \mathrm{d}\mathcal{H}^{n-1} \geq \frac{n^{n-1}}{2} \left[\frac{\mathrm{W}_2^{n-1}}{\left(|\mathrm{bd}\,K| + \frac{\xi_1}{\mathrm{W}_2} \right)^{n-2}} + \frac{\mathrm{W}_2^{n-1}}{\left(|\mathrm{bd}\,K| - \frac{\xi_1}{\mathrm{W}_2} \right)^{n-2}} \right],
$$

$$
\int_{\mathrm{bd}\,K} \frac{1}{H} \mathrm{d}\mathcal{H}^{n-1} \geq \frac{1}{n} \frac{|\mathrm{bd}\,K|^2 \mathrm{W}_2}{\mathrm{W}_2^2 - \frac{\xi_1^2}{|\mathrm{bd}\,K|^2}} \quad \text{and}
$$

$$
\int_{\mathrm{bd}\,K} H^2 \mathrm{d}\mathcal{H}^{n-1} \geq n^2 \frac{|\mathrm{bd}\,K|\mathrm{W}_2^2}{|\mathrm{bd}\,K|^2 - \frac{\xi_1^2}{\mathrm{W}_2^2}}.
$$

Equality holds in all inequalities if $K = B_2^n$ *(up to dilations).*

The first and second bounds, as well as the ones obtained with the tighter inequalities given in Theorems 3.2 and 3.3, improve the known ones for convex hypersurfaces given by (1.1) and (1.2) (see Remark 2).

Corollary 1.2. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then,*

$$\int_{\mathrm{bd}\,K} \kappa^2 \, \mathrm{d}\mathcal{H}^{n-1} \geq n^2 \frac{|B_2^n|^2 |\,\mathrm{bd}\,K|}{|\,\mathrm{bd}\,K|^2 - \frac{\xi_{n-1}^2}{|B_2^n|^2}},$$

$$\int_{\mathrm{bd}\,K} \frac{1}{\kappa} \, \mathrm{d}\mathcal{H}^{n-1} \geq \frac{1}{n} \frac{|B_2^n| |\,\mathrm{bd}\,K|^2}{|B_2^n|^2 - \frac{\xi_{n-1}^2}{|\,\mathrm{bd}\,K|^2}}.$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

These higher dimensional extensions will be consequences of a very general result for functions defined on a general probability space, which will be obtained following the ideas in [7]. More precisely, we will prove the following proposition. Here, as usual in the literature, $\mathbb{E}\rho$ will denote the expectation of ρ, $\mathrm{Cov}(\rho, h) = \mathbb{E}h\rho - \mathbb{E}h\mathbb{E}\rho$ the covariance of ρ and h, and $\|\cdot\|_\infty$ the sup-norm, i.e., $\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$.

Proposition 1.1. *Let $(\Omega, \mathbb{P})$ be a probability space such that, for any $A \subseteq \Omega$ with positive probability and any $0 \leq p \leq \mathbb{P}(A)$, there exists $B \subseteq A$ with $\mathbb{P}(B) = p$. Let $\rho, h : \Omega \longrightarrow \mathbb{R}$, with $\rho \in L^1(\Omega)$ and $h \in L^\infty(\Omega)$. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the expressions below are defined, we have*

$$\mathbb{E}(F \circ \rho) \geq \frac{F\left(\mathbb{E}\rho + \frac{\mathrm{Cov}(\rho, h)}{\|h - \mathbb{E}h\|_\infty}\right) + F\left(\mathbb{E}\rho - \frac{\mathrm{Cov}(\rho, h)}{\|h - \mathbb{E}h\|_\infty}\right)}{2}.$$

The paper is organized as follows. In Section 2 we study the probabilistic type results that will be the key to the later development; in particular, we prove Proposition 1.1. Then Section 3 is devoted to provide tight estimates for the integral of any convex function of the i-th mean curvatures H_i (Theorem 3.1), which will allow to get, as particular cases, lower bounds for the integral of any power of H_i and $1/H_i$ (Theorems 3.2 and 3.3). Other consequences, such as an estimate for the entropy of H_i, are also obtained.

Analogous results to the ones of the H_i's, but for the elementary symmetric functions of the radii of curvature of a C_+^2 convex body, are proved in Section 4 (Theorem 4.1); additional considerations regarding the possibility of removing the inner and outer radii from the bounds are also made. Then, Section 5 is devoted to relate the integral of the symmetric functions of the radii of curvature, to the roots of the Steiner polynomial of the convex body, as well as to consider particular families of bodies like constant width sets. Finally, in Section 6, we provide tight estimates for the integral of any convex function of (powers of) the radial function of a convex body. In this case, the bounds will be given in terms of the so-called dual quermaßintegrals.

2. Some probabilistic type results

In this section we show the probabilistic results (Proposition 1.1 and, as a consequence, Proposition 2.1) from which all theorems will be obtained.

We start with the proof of Proposition 1.1.

Proof of Proposition 1.1. Without loss of generality we can assume that $\mathrm{Cov}(\rho, h) \leq 0$; otherwise we just change h by $-h$.

Let m be the median of ρ and let $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \backslash \Omega_1$ be such that $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1/2$ and

$$\{\omega \in \Omega : \rho(\omega) > m\} \subseteq \Omega_1 \subseteq \{\omega \in \Omega : \rho(\omega) \geq m\},$$
$$\{\omega \in \Omega : \rho(\omega) < m\} \subseteq \Omega_2 \subseteq \{\omega \in \Omega : \rho(\omega) \leq m\}.$$

The fact that if $\mathbb{P}\big(\{\omega \in \Omega : \rho(\omega) = m\}\big)$ is positive then we can divide it into two disjoint sets of appropriate measure, guarantees that such Ω_1 exists. Let

$$\rho_1 = 2 \int_{\Omega_1} \rho(\omega) \, d\mathbb{P}(\omega) \quad \text{and} \quad \rho_2 = 2 \int_{\Omega_2} \rho(\omega) \, d\mathbb{P}(\omega).$$

Since $\rho_1 + \rho_2 = 2 \mathbb{E}\rho$, we can write

$$(2.1) \qquad \rho_1 = \mathbb{E}\rho + b \quad \text{and} \quad \rho_2 = \mathbb{E}\rho - b$$

for some $b \geq 0$. First, we are going to prove that

$$(2.2) \qquad \frac{\big|\mathrm{Cov}(\rho, h)\big|}{\|h - \mathbb{E}h\|_\infty} \leq b.$$

Indeed, since

$$-\|h - \mathbb{E}h\|_\infty \leq \mathbb{E}h - h(\omega) \leq \|h - \mathbb{E}h\|_\infty$$

for every $\omega \in \Omega$ and since $\rho(\omega) \geq m$ if $\omega \in \Omega_1$ and $\rho(\omega) \leq m$ if $\omega \in \Omega_2$, then we have that

$$\int_{\Omega_1} \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega) - m\big) \, d\mathbb{P}(\omega) \leq \frac{1}{2}\|h - \mathbb{E}h\|_\infty(\rho_1 - m)$$

and

$$\int_{\Omega_2} \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega) - m\big) \, d\mathbb{P}(\omega) \leq -\frac{1}{2}\|h - \mathbb{E}h\|_\infty(\rho_2 - m).$$

Adding both integrals and using (2.1) we get

$$\mathbb{E}\big((\mathbb{E}h - h)(\rho - m)\big) = \int_\Omega \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega) - m\big) \, d\mathbb{P}(\omega)$$
$$\leq \frac{1}{2}\|h - \mathbb{E}h\|_\infty(\rho_1 - \rho_2) = \|h - \mathbb{E}h\|_\infty b,$$

and since

$$\mathbb{E}\big((\mathbb{E}h - h)(\rho - m)\big) = \mathbb{E}h\mathbb{E}\rho - \mathbb{E}h\rho = -\mathrm{Cov}(\rho, h),$$

we obtain the required bound (2.2).

Now, since F is convex, Jensen's inequality (see e.g. [16, p. 20]) yields

$$F(\rho_1) \leq 2 \int_{\Omega_1} (F \circ \rho)(\omega) \, d\mathbb{P}(\omega) \quad \text{and} \quad F(\rho_2) \leq 2 \int_{\Omega_2} (F \circ \rho)(\omega) \, d\mathbb{P}(\omega),$$

which, together with (2.1) and (2.2), implies the result:

$$
\begin{aligned}
\mathbb{E}(F \circ \rho) &= \int_{\Omega_1} (F \circ \rho)(\omega) \, d\mathbb{P}(\omega) + \int_{\Omega_2} (F \circ \rho)(\omega) \, d\mathbb{P}(\omega) \\
&\geq \frac{F(\rho_1) + F(\rho_2)}{2} = \frac{F(\mathbb{E}\rho + b) + F(\mathbb{E}\rho - b)}{2} \\
&\geq \frac{F\left(\mathbb{E}\rho + \frac{\mathrm{Cov}(\rho,h)}{\|h - \mathbb{E}h\|_\infty}\right) + F\left(\mathbb{E}\rho - \frac{\mathrm{Cov}(\rho,h)}{\|h - \mathbb{E}h\|_\infty}\right)}{2}.
\end{aligned}
$$
□

An interesting particular case is obtained when $F(x) = 1/x$:

Corollary 2.1. *Let $(\Omega, \mathbb{P})$ be a probability space such that, for any $A \subseteq \Omega$ with positive probability and any $0 \leq p \leq \mathbb{P}(A)$, there exists $B \subseteq A$ with $\mathbb{P}(B) = p$. Let $\rho, h : \Omega \longrightarrow \mathbb{R}$, with $\rho \in L^1(\Omega)$ and $h \in L^\infty(\Omega)$. Then,*

$$
\mathbb{E}\frac{1}{\rho} \geq \frac{\mathbb{E}\rho}{(\mathbb{E}\rho)^2 - \frac{\mathrm{Cov}(\rho,h)^2}{\|h - \mathbb{E}h\|_\infty^2}}.
$$

If we express the probability measure by means of a density with respect to another (not necessarily a probability) measure μ, we immediately get the following result.

Proposition 2.1. *Let (Ω, μ) be a measure space and let $g : \Omega \longrightarrow \mathbb{R}$ be a positive integrable function such that $\int_\Omega g \, d\mu = 1$, and for any $A \subseteq \Omega$ with $\int_A g \, d\mu > 0$ and any $0 \leq p \leq \int_A g \, d\mu$, there exists $B \subseteq A$ with $\int_B g \, d\mu = p$. Let $\rho, h : \Omega \longrightarrow \mathbb{R}$ be integrable functions with $h \in L^\infty(\Omega)$. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the expressions below are defined, we have*

$$
\int_\Omega (F \circ \rho) g \, d\mu \geq \frac{F\left(\int_\Omega \rho g \, d\mu + \eta(\rho, h, g)\right) + F\left(\int_\Omega \rho g \, d\mu - \eta(\rho, h, g)\right)}{2},
$$

where

$$
\eta(\rho, h, g) = \frac{\int_\Omega \rho h g \, d\mu - \left(\int_\Omega \rho g \, d\mu\right)\left(\int_\Omega h g \, d\mu\right)}{\left\|h - \int_\Omega h g \, d\mu\right\|_\infty}.
$$

Rewriting the above proposition for some particular functions, special inequalities can be obtained. For instance, we can take $\rho = 1/g$ and, either $F(x) = 1/x^\alpha$ or $F(x) = x^{\alpha+1}$ with $\alpha \geq 0$:

Corollary 2.2. *Let (Ω, μ) be a measure space and let $g : \Omega \to \mathbb{R}$ be a positive integrable function with $\int_\Omega g \, d\mu = 1$, $1/g$ integrable and such that, for any $A \subseteq \Omega$ with $\int_A g \, d\mu > 0$ and any $0 \leq p \leq \int_A g \, d\mu$, there exists $B \subseteq A$ with $\int_B g \, d\mu = p$. Let $h : \Omega \longrightarrow \mathbb{R}$ be an integrable function, $h \in L^\infty(\Omega)$, and let*

$$
(2.3) \qquad \eta(h, g) = \frac{\int_\Omega h \, d\mu - \mu(\Omega) \int_\Omega h g \, d\mu}{\left\|h - \int_\Omega h g \, d\mu\right\|_\infty}.
$$

Then, for any $\alpha \geq 0$,

$$(2.4) \qquad \int_\Omega g^{\alpha+1} \mathrm{d}\mu \geq \frac{1}{2} \left(\frac{1}{\left(\mu(\Omega) + \eta(h,g)\right)^\alpha} + \frac{1}{\left(\mu(\Omega) - \eta(h,g)\right)^\alpha} \right)$$

and

$$(2.5) \qquad \int_\Omega \frac{1}{g^\alpha} \mathrm{d}\mu \geq \frac{\left(\mu(\Omega) + \eta(h,g)\right)^{\alpha+1} + \left(\mu(\Omega) - \eta(h,g)\right)^{\alpha+1}}{2}$$

In particular, taking $\alpha = 1$ in (2.4) and in (2.5), we get

$$\int_\Omega g^2 \mathrm{d}\mu \geq \frac{\mu(\Omega)}{\mu(\Omega)^2 - \eta(h,g)^2} \qquad \text{and} \qquad \int_\Omega \frac{1}{g} \mathrm{d}\mu \geq \mu(\Omega)^2 + \eta(h,g)^2,$$

respectively.

It is also of particular interest to obtain an estimate for the entropy of g, i.e., to consider $F(x) = -\log x$ and again $\rho = 1/g$.

Corollary 2.3. *Let (Ω, μ) be a measure space and let $g : \Omega \longrightarrow \mathbb{R}$ be an integrable function with $\int_\Omega g \, \mathrm{d}\mu = 1$, $1/g$ integrable and such that, for any $A \subseteq \Omega$ with $\int_A g \, \mathrm{d}\mu > 0$ and any $0 \leq p \leq \int_A g \, \mathrm{d}\mu$, there exists $B \subseteq A$ with $\int_B g \, \mathrm{d}\mu = p$. Let $h : \Omega \longrightarrow \mathbb{R}$ be an integrable function, $h \in L^\infty(\Omega)$. Then*

$$\int_\Omega g \log g \, \mathrm{d}\mu \geq -\frac{1}{2} \log\left(\mu(\Omega)^2 - \eta(h,g)^2\right),$$

where $\eta(h,g)$ is defined as in (2.3).

3. Mean curvatures of hypersurfaces

For a convex body $K \in \mathcal{K}_0^n$ and a non-negative real number λ, the volume of the Minkowski sum (vectorial addition) $K + \lambda B_2^n$ is expressed as a polynomial of degree n in λ, namely,

$$(3.1) \qquad |K + \lambda B_2^n| = \sum_{i=0}^n \binom{n}{i} \mathrm{W}_i(K) \lambda^i,$$

which is called the (classical) *Steiner formula* of K (see [17]). The coefficients $\mathrm{W}_i(K)$ are the *quermaßintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [16, s. 5.1]. In particular, $\mathrm{W}_0(K) = |K|$ (the area $\mathrm{A}(K)$ in the planar case), $n\mathrm{W}_1(K) = |\operatorname{bd} K|$ (the perimeter $\mathrm{p}(K)$ in the plane) and $2\mathrm{W}_{n-1}/|B_2^n| = \mathrm{b}(K)$ is the mean width of K. Moreover, $\mathrm{W}_n(K) = |B_2^n|$, which takes the value

$$|B_2^n| = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

where Γ is the gamma function. Then, denoting by B the beta function, it is easy to check the one-dimensional recursion formula (see e.g. [18, s. 5.3])

$$(3.2) \qquad \frac{|B_2^n|}{|B_2^{n-1}|} = \frac{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} = \mathrm{B}\left(\frac{1}{2}, \frac{n+1}{2}\right) = 2 \int_0^1 \left(1 - x^2\right)^{(n-1)/2} \mathrm{d}x.$$

From now on, for the sake of brevity, we will write $W_i = W_i(K)$, $i = 0, \ldots, n$, and analogously for all radii, if the distinction of the body is not needed.

Further, for $1 \leq j \leq k \leq n-1$, $0 \leq l \leq n-1$, let

$$
(3.3) \quad
\begin{aligned}
\eta_{j,k,l} &= \frac{W_j W_k - W_{j-1} W_{k+1}}{W_{l+1}^2 \min_{x \in K} \max \left\{ \bar{R}(K-x) - \dfrac{W_l}{W_{l+1}}, \dfrac{W_l}{W_{l+1}} - \bar{r}(K-x) \right\}} \\
&\qquad \text{if } K \neq B_2^n(r) \text{ for any } r > 0, \\
\eta_{j,k,l} &= 0 \quad \text{if } K = B_2^n(r) \text{ for some } r > 0.
\end{aligned}
$$

We observe that since $\bar{r}B_2^n \subseteq K$ and $K \subseteq \bar{R}B_2^n$, the inequalities

$$(3.4) \qquad\qquad \bar{r}W_{i+1} \leq W_i \leq \bar{R}W_{i+1},$$

$i = 0, \ldots, n-1$, are a direct consequence of the monotonicity of the mixed volumes (cf. e.g. [16, p. 282]); hence the maximum in (3.3) is always positive (quermaßintegrals are translation invariant functionals). This, together with the inequalities

$$(3.5) \qquad W_j W_k \geq W_{j-1} W_{k+1}, \quad 1 \leq j \leq k \leq n-1,$$

which are particular cases of the Aleksandrov-Fenchel inequality (see e.g. [16, s. 7.3]), ensures that $\eta_{j,k,l} \geq 0$.

Remark 1. *At this point we would like to observe that the point $x \in K$ for which the minimum in (3.3) is attained and c_K do not necessarily coincide, and thus, we cannot just assume that c_K is the origin in all the results. Indeed, taking the triangle*

$$T = \mathrm{conv}\left\{ (0,0)^{\mathsf{T}}, (1,0)^{\mathsf{T}}, \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)^{\mathsf{T}} \right\},$$

it is easy to see that $c_T = \left(1/2, \left(2 - \sqrt{3} \right)/2 \right)^{\mathsf{T}}$, $\bar{R}(T - c_T) = \left(\sqrt{6} - \sqrt{2} \right)/2$ and $\bar{r}(T - c_T) = \left(2 - \sqrt{3} \right)/2$, which yields

$$\max\left\{ \bar{R}(T - c_T) - \frac{\mathrm{p}}{2\pi}, \frac{\mathrm{p}}{2\pi} - \bar{r}(T - c_T) \right\} = \frac{\mathrm{p}}{2\pi} - \bar{r}(T - c_T) = 0.2667\ldots$$

However, there are translation points $x \in T$ for which the corresponding maximum is smaller: for instance, if we take $x = \left(3/5, 3(2 - \sqrt{3})/5 \right)^{\mathsf{T}}$, it can be checked that

$$\max\left\{ \bar{R}(T - x) - \frac{\mathrm{p}}{2\pi}, \frac{\mathrm{p}}{2\pi} - \bar{r}(T - x) \right\} = \frac{\mathrm{p}}{2\pi} - \bar{r}(T - x) = 0.2399\ldots$$

The C_+^2 case follows by approximation.

Throughout the paper we will denote by

$$\sigma_i(x_1, \ldots, x_m) = \sum_{1 \leq j_1 < \cdots < j_i \leq m} x_{j_1} \cdot \ldots \cdot x_{j_i}$$

the i-th elementary symmetric function of $x_1, \ldots, x_m \in \mathbb{R}$, $1 \leq i \leq m$, setting $\sigma_0(x_1, \ldots, x_m) = 1$. Furthermore, we will use $\mathrm{s}_i(x_1, \ldots, x_m)$ for the normalized i-th elementary symmetric function, i.e.,

$$\mathrm{s}_i(x_1, \ldots, x_m) = \binom{m}{i}^{-1} \sum_{1 \leq j_1 < \cdots < j_i \leq m} x_{j_1} \cdot \ldots \cdot x_{j_i},$$

with $\mathrm{s}_0(x_1, \ldots, x_m) = 1$.

If $K \in \mathcal{K}_0^n$ is of class C_+^2, then its boundary $\mathrm{bd}\, K$ is a (compact) hypersurface of $\mathbb{R}^n$, and we can consider the $n-1$ principal curvatures $k_1, \ldots, k_{n-1}$ of $\mathrm{bd}\, K$. As usual in the literature, by

$$H_i = \mathrm{s}_i(k_1, \ldots, k_{n-1}), \quad i = 1, \ldots, n-1,$$

we denote the *i-th mean curvature*, setting $H_0 = 1$, and we refer to $H = H_1$ as the classical *mean curvature* and to $\kappa = H_{n-1}$ as the *Gauss-Kronecker curvature*. Then (see e.g. [16, pp. 296–297]) it is known that, for $i = 1, \ldots, n$,

$$(3.6) \qquad \mathrm{W}_i = \frac{1}{n} \int_{\mathrm{bd}\, K} H_{i-1} \, \mathrm{d}\mathcal{H}^{n-1} = \frac{1}{n} \int_{\mathrm{bd}\, K} q_K H_i \, \mathrm{d}\mathcal{H}^{n-1},$$

where $q_K = h_K \circ \nu_K$ and $h_K(u) = \sup_{x \in K} \langle x, u \rangle$, $u \in \mathbb{R}^n$, is the support function of K (see e.g. [16, s. 1.7]). We observe that the volume

$$|K| = \mathrm{W}_0 = \frac{1}{n} \int_{\mathrm{bd}\, K} q_K H_0 \, \mathrm{d}\mathcal{H}^{n-1} = \frac{1}{n} \int_{\mathrm{bd}\, K} q_K \, \mathrm{d}\mathcal{H}^{n-1}.$$

First we show the following general result for an arbitrary convex function.

Theorem 3.1. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $0 \leq i \leq n-1$,*

$$(3.7) \quad \int_{\mathrm{bd}\, K} (F \circ H_i) \, \mathrm{d}\mathcal{H}^{n-1} \geq n\mathrm{W}_1 \frac{F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1} + \eta_{1,i,0}\right) + F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1} - \eta_{1,i,0}\right)}{2},$$

(3.8)

$$\int_{\mathrm{bd}\, K} \left(F \circ \frac{1}{H_i}\right) H_i \, \mathrm{d}\mathcal{H}^{n-1} \geq n\mathrm{W}_{i+1} \frac{F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}} + \eta_{1,i,i}\right) + F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}} - \eta_{1,i,i}\right)}{2}.$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

Proof. In order to get inequality (3.7), we consider the probability space $\left(\mathrm{bd}\, K, \mathcal{H}^{n-1}/(n\mathrm{W}_1)\right)$ and apply Proposition 1.1 to the functions $\rho = H_i$ and $h = q_K$. Then, using the identities in (3.6), we get $\mathbb{E}\rho = \mathrm{W}_{i+1}/\mathrm{W}_1$, $\mathbb{E}h = \mathrm{W}_0/\mathrm{W}_1$ and

$$\mathrm{Cov}(\rho, h) = \mathbb{E}h\rho - \mathbb{E}h\mathbb{E}\rho = \frac{\mathrm{W}_i \mathrm{W}_1 - \mathrm{W}_0 \mathrm{W}_{i+1}}{\mathrm{W}_1^2}.$$

Moreover,

$$\|h - \mathbb{E}h\|_\infty = \sup\left\{\left|q_K(x) - \frac{\mathrm{W}_0}{\mathrm{W}_1}\right| : x \in \mathrm{bd}\, K\right\} = \max\left\{\bar{\mathrm{R}} - \frac{\mathrm{W}_0}{\mathrm{W}_1}, \frac{\mathrm{W}_0}{\mathrm{W}_1} - \bar{\mathrm{r}}\right\},$$

and since the functionals H_j, W_j are translation invariant, the greatest possible lower bound for $\int_{\mathrm{bd}\,K}(F\circ H_i)\,\mathrm{d}\mathcal{H}^{n-1}$ will be obtained for the translation of K such that the above maximum is minimal. Altogether gives (3.7).

Inequality (3.8) is obtained analogously, but now as a consequence of Proposition 2.1 for $\rho = 1/H_i$, $h = q_K$ and $g = H_i/(nW_{i+1})$; we notice that, by (3.6), $\int_{\mathrm{bd}\,K} g\,\mathrm{d}\mathcal{H}^{n-1} = 1$.

Equality trivially holds for $K = B_2^n$ (up to dilations) just noticing that $W_i(B_2^n) = |B_2^n|$ for all $i = 0,\dots,n$ and $|\mathbb{S}^{n-1}| = n|B_2^n|$. $\qquad\square$

As our main aim is to get bounds for the integral of powered mean curvatures, this may be achieved applying Theorem 3.1 to the functions $F(x) = x^{\alpha+1}$ or $F(x) = 1/x^{\alpha}$, $\alpha \geq 0$. Thus, two different results can be obtained, providing different bounds for the same integrals:

Theorem 3.2. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then, for all $0 \leq i \leq n-1$ and any $\alpha \geq 0$,*

(3.9)
$$\int_{\mathrm{bd}\,K} H_i^{\alpha+1}\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{(W_{i+1} + W_1\,\eta_{1,i,0})^{\alpha+1}}{W_1^{\alpha}} + \frac{(W_{i+1} - W_1\,\eta_{1,i,0})^{\alpha+1}}{W_1^{\alpha}}\right],$$

(3.10)
$$\int_{\mathrm{bd}\,K} \frac{1}{H_i^{\alpha}}\,\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{(W_1 + W_{i+1}\,\eta_{1,i,i})^{\alpha+1}}{W_{i+1}^{\alpha}} + \frac{(W_1 - W_{i+1}\,\eta_{1,i,i})^{\alpha+1}}{W_{i+1}^{\alpha}}\right].$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

Theorem 3.3. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then, for all $0 \leq i \leq n-1$ and any $\alpha \geq 0$,*

(3.11)
$$\int_{\mathrm{bd}\,K} \frac{1}{H_i^{\alpha}}\,\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{W_1^{\alpha+1}}{(W_{i+1} + W_1\,\eta_{1,i,0})^{\alpha}} + \frac{W_1^{\alpha+1}}{(W_{i+1} - W_1\,\eta_{1,i,0})^{\alpha}}\right],$$

(3.12)
$$\int_{\mathrm{bd}\,K} H_i^{\alpha+1}\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{W_{i+1}^{\alpha+1}}{(W_1 + W_{i+1}\,\eta_{1,i,i})^{\alpha}} + \frac{W_{i+1}^{\alpha+1}}{(W_1 - W_{i+1}\,\eta_{1,i,i})^{\alpha}}\right].$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

The lower bounds for $\int_{\mathrm{bd}\,K} H_i^{\alpha+1}\mathrm{d}\mathcal{H}^{n-1}$ given by (3.9) and (3.12) cannot be compared; indeed, depending on the set, each one of them may be tighter (the same occurs in the case of $\int_{\mathrm{bd}\,K} 1/H_i^{\alpha}\mathrm{d}\mathcal{H}^{n-1}$). For instance, it can be checked that if K is a convex body such that

$$\min_{x\in K}\max\left\{\bar{R}(K-x) - \frac{W_0}{W_1}, \frac{W_0}{W_1} - \bar{r}(K-x)\right\}$$

$$\geq \min_{x\in K}\max\left\{\bar{R}(K-x) - \frac{W_i}{W_{i+1}}, \frac{W_i}{W_{i+1}} - \bar{r}(K-x)\right\},$$

then (3.12) provides a better bound. However, weaker bounds, which are obtained from Theorems 3.2 and 3.3, can be compared. We see it in the following, proving Theorem 1.1.

Proof of Theorem 1.1. First we observe that the fact that the functions $x^{\alpha+1}$ and $1/x^{\alpha}$ are convex, allows $\eta_{1,i,0}$ and $\eta_{1,i,i}$ to be replaced by smaller numbers in Theorems 3.2 and 3.3: by (3.4) and (3.5) we immediately get

$$\eta_{1,i,0} \geq \frac{\mathrm{W}_i\mathrm{W}_1 - \mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1^2 \min_{x\in K}\{\bar{\mathrm{R}}(K-x) - \bar{\mathrm{r}}(K-x)\}} = \frac{\xi_i}{n\mathrm{W}_1^2} \geq 0 \quad \text{and}$$

$$\eta_{1,i,i} \geq \frac{\mathrm{W}_i\mathrm{W}_1 - \mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_{i+1}^2 \min_{x\in K}\{\bar{\mathrm{R}}(K-x) - \bar{\mathrm{r}}(K-x)\}} = \frac{\xi_i}{n\mathrm{W}_{i+1}^2} \geq 0$$

(see the definition of ξ_i in Theorem 1.1), and thus (3.12) and (3.9) yield

$$(3.13) \quad \int_{\mathrm{bd}\,K} H_i^{\alpha+1}\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(\mathrm{W}_1 + \frac{\xi_i}{n\mathrm{W}_{i+1}}\right)^{\alpha}} + \frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(\mathrm{W}_1 - \frac{\xi_i}{n\mathrm{W}_{i+1}}\right)^{\alpha}}\right],$$

$$(3.14) \quad \int_{\mathrm{bd}\,K} H_i^{\alpha+1}\mathrm{d}\mathcal{H}^{n-1} \geq \frac{n}{2}\left[\frac{\left(\mathrm{W}_{i+1} + \frac{\xi_i}{n\mathrm{W}_1}\right)^{\alpha+1}}{\mathrm{W}_1^{\alpha}} + \frac{\left(\mathrm{W}_{i+1} - \frac{\xi_i}{n\mathrm{W}_1}\right)^{\alpha+1}}{\mathrm{W}_1^{\alpha}}\right],$$

respectively. But it can be checked that the lower bound in (3.13) is greater than the one in (3.14). Indeed, easy computations allow to see that this fact is equivalent to the inequality

$$(3.15) \qquad \frac{1}{(1+x)^{\alpha}} + \frac{1}{(1-x)^{\alpha}} \geq (1+x)^{\alpha+1} + (1-x)^{\alpha+1}$$

for any $\alpha \geq 0$, where

$$x = \frac{\xi_i}{n\mathrm{W}_1\mathrm{W}_{i+1}} = \frac{\mathrm{W}_i\mathrm{W}_1 - \mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1\mathrm{W}_{i+1}\left(\bar{\mathrm{R}}(K-c_K) - \bar{\mathrm{r}}(K-c_K)\right)}.$$

Using again (3.4) and (3.5), we get that

$$0 \leq \mathrm{W}_i\mathrm{W}_1 - \mathrm{W}_0\mathrm{W}_{i+1} \leq \mathrm{W}_1\mathrm{W}_{i+1}\left(\bar{\mathrm{R}}(K-c_K) - \bar{\mathrm{r}}(K-c_K)\right),$$

and therefore, $0 \leq x \leq 1$. Thus, (3.15) is equivalent to see that

$$f(x) = \frac{\frac{1}{(1+x)^{\alpha}} + \frac{1}{(1-x)^{\alpha}}}{(1+x)^{\alpha+1} + (1-x)^{\alpha+1}} \geq 1$$

for all $x \in [0,1]$ and any $\alpha \geq 0$. A direct computation shows that the numerator in

$$f'(x) = \frac{(1-x)^{2\alpha+1}\left[(2\alpha+1)x + 1\right] - (1+x)^{2\alpha+1}\left[1 - (2\alpha+1)x\right]}{(1-x)^{\alpha+1}(1+x)^{\alpha+1}\left[(1-x)^{\alpha+1} + (1+x)^{\alpha+1}\right]^2}$$

is an increasing function in $x \in [0,1]$, and so non-negative, which ensures that $f(x)$ itself is also increasing and hence $f(x) \geq f(0) = 1$, as required. Analogous computations show the lower bound for $\int_{\mathrm{bd}\,K} 1/H_i^{\alpha}\mathrm{d}\mathcal{H}^{n-1}$.   □

Remark 2. *We notice that in the case $i = 1$ (mean curvature), the estimates for $\int_{\mathrm{bd}\,K} H^{n-1} \mathrm{d}\mathcal{H}^{n-1}$ and $\int_{\mathrm{bd}\,K} 1/H\,\mathrm{d}\mathcal{H}^{n-1}$ which are obtained from Theorem 1.1 when, respectively, $\alpha = n - 2$ and $\alpha = 1$ (see Corollary 1.1), improve the known ones for convex hypersurfaces given by (1.1) and (1.2). Indeed, on the one hand,*

$$\frac{1}{n} \frac{|\,\mathrm{bd}\,K|^2 \mathrm{W}_2}{\mathrm{W}_2^2 - \frac{\xi_1^2}{|\,\mathrm{bd}\,K|^2}} \geq \frac{1}{n} \frac{|\,\mathrm{bd}\,K|^2 \mathrm{W}_2}{\mathrm{W}_2^2} = n \frac{\mathrm{W}_1^2}{\mathrm{W}_2} \geq n|K|$$

because of the Aleksandrov-Fenchel inequality (3.5) for $j = k = 1$. On the other hand, since the function $1/x^{n-2}$ is convex, then

$$\frac{n^{n-1}}{2} \left[\frac{\mathrm{W}_2^{n-1}}{\left(|\,\mathrm{bd}\,K| + \frac{\xi_1}{\mathrm{W}_2}\right)^{n-2}} + \frac{\mathrm{W}_2^{n-1}}{\left(|\,\mathrm{bd}\,K| - \frac{\xi_1}{\mathrm{W}_2}\right)^{n-2}} \right] \geq \frac{n^{n-1}}{2} \frac{2\mathrm{W}_2^{n-1}}{|\,\mathrm{bd}\,K|^{n-2}}$$

$$= n \frac{\mathrm{W}_2^{n-1}}{\mathrm{W}_1^{n-2}} \geq n|B_2^n|,$$

where the last inequality follows from the known relations $\mathrm{W}_j^{k-i} \geq \mathrm{W}_i^{k-j} \mathrm{W}_k^{j-i}$, $0 \leq i < j < k \leq n$, which are also consequences of the Aleksandrov-Fenchel inequality (see e.g. [16, (7.66)]). Similar considerations can be made regarding Theorems 3.2 and 3.3.

It may also have interest to obtain an estimate for the entropy of the i-th mean curvatures. We do it in the following result.

Corollary 3.1. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then, for all $0 \leq i \leq n - 1$,*

$$\int_{\mathrm{bd}\,K} H_i \log H_i \, \mathrm{d}\mathcal{H}^{n-1} \geq -\frac{n}{2} \mathrm{W}_{i+1} \log \left(\frac{\mathrm{W}_1^2}{\mathrm{W}_{i+1}^2} - \eta_{1,i,i}^2 \right).$$

Equality holds if $K = B_2^n$ (up to dilations).

Proof. It is a direct consequence of Theorem 3.1, just taking the convex function $F(x) = \log(1/x)$ in inequality (3.8). $\square$

4. RADII OF CURVATURE OF CONVEX BODIES

If $K \in \mathcal{K}_0^n$ is of class C_+^2, we can consider the $n - 1$ principal radii of curvature $r_1, \ldots, r_{n-1}$ of K at $u \in \mathbb{S}^{n-1}$, i.e., the eigenvalues of the reverse Weingarten map (see e.g. [16, p. 116] for a detailed explanation). Then, for $i = 1, \ldots, n - 1$, we denote by

$$\varrho_i = \mathrm{s}_i \left(r_1, \ldots, r_{n-1} \right)$$

the i-th normalized elementary symmetric function of the principal radii of curvature, with $\varrho_0 = 1$, and we observe that, properly ordering the indices,

$$r_i(u) = \frac{1}{k_i\left(x_K(u)\right)},$$

$i = 1, \ldots, n-1$, where $x_K(u) \in \operatorname{bd} K$ is the unique point of the boundary at which u is the outer normal vector. Moreover, for all $u \in \mathbb{S}^{n-1}$, $x \in \operatorname{bd} K$,

$$\varrho_i(u) = \frac{H_{n-i-1}}{H_{n-1}}\big(x_K(u)\big) \quad \text{and} \quad H_i(x) = \frac{\varrho_{n-i-1}}{\varrho_{n-1}}\big(\nu_K(x)\big),$$

and then (see e.g. [16, pp. 296–297])

$$(4.1) \qquad \mathrm{W}_i = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \varrho_{n-i}\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K \varrho_{n-i-1}\,\mathrm{d}\mathcal{H}^{n-1},$$

$i = 0, \ldots, n-1$; the right-hand side identities in (3.6) and (4.1) are usually known in the literature as Minkowski's integral formulae.

An analogous result to Theorem 3.1, now replacing the i-th mean curvatures by ϱ_i, can be obtained. For the sake of brevity we write (see (3.3)) $\bar{\eta}_{j,k,l} = \eta_{n-j,n-k,n-l}$.

Theorem 4.1. *Let* $K \in \mathcal{K}_0^n$ *be of class* C_+^2. *Then, for any convex function* $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ *where all the quantities are defined, and all* $0 \le i \le n-1$,

$$\int_{\mathbb{S}^{n-1}} (F \circ \varrho_i)\,\mathrm{d}\mathcal{H}^{n-1} \ge n|B_2^n| \frac{F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} + \bar{\eta}_{i,1,1}\right) + F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} - \bar{\eta}_{i,1,1}\right)}{2},$$

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \frac{1}{\varrho_i}\right) \varrho_i\,\mathrm{d}\mathcal{H}^{n-1} \ge n\mathrm{W}_{n-i} \frac{F\left(\frac{|B_2^n|}{\mathrm{W}_{n-i}} + \bar{\eta}_{i,1,i+1}\right) + F\left(\frac{|B_2^n|}{\mathrm{W}_{n-i}} - \bar{\eta}_{i,1,i+1}\right)}{2}.$$

Equality holds in both inequalities if $K = B_2^n$ *(up to dilations).*

Proof. The first inequality is obtained applying Proposition 1.1 to the probability space $\big(\mathbb{S}^{n-1}, \mathcal{H}^{n-1}/(n|B_2^n|)\big)$ and the functions $\rho = \varrho_i$ and $h = h_K$. Then, using (4.1), we get $\mathbb{E}\rho = \mathrm{W}_{n-i}/|B_2^n|$, $\mathbb{E}h = \mathrm{W}_{n-1}/|B_2^n|$,

$$\operatorname{Cov}(\rho, h) = \frac{\mathrm{W}_{n-i-1}|B_2^n| - \mathrm{W}_{n-i}\mathrm{W}_{n-1}}{|B_2^n|^2}$$

and

$$\|h - \mathbb{E}h\|_\infty = \sup\left\{\left|h_K(u) - \frac{\mathrm{W}_{n-1}}{|B_2^n|}\right| : u \in \mathbb{S}^{n-1}\right\}$$

$$= \max\left\{\bar{\mathrm{R}} - \frac{\mathrm{W}_{n-1}}{|B_2^n|}, \frac{\mathrm{W}_{n-1}}{|B_2^n|} - \bar{\mathrm{r}}\right\}.$$

Altogether, and taking into account that all bounds obtained for all possible translations of K are valid, shows the first inequality.

Second inequality is obtained analogously, but now as a consequence of Proposition 2.1 for $\rho = 1/\varrho_i$, $h = h_K$ and $g = \varrho_i/(n\mathrm{W}_{n-i})$; we notice that, by (4.1), $\int_{\mathbb{S}^{n-1}} g\,\mathrm{d}\mathcal{H}^{n-1} = 1$. The equality case is trivial. $\qquad\square$

At this point we would like to mention that analogous results to Theorem 3.2, Theorem 3.3 and Corollary 3.1, for ϱ_i, can be also proved replacing the convex function F in Theorem 4.1 by the suitable ones. And again, it turns out that it will be not possible to compare the two lower bounds that

can be obtained for $\int_{\mathbb{S}^{n-1}} \varrho_i^{\alpha+1} \mathrm{d}\mathcal{H}^{n-1}$, because, depending on the set, each of them might be tighter (analogously for $\int_{\mathbb{S}^{n-1}} 1/\varrho_i^\alpha \, \mathrm{d}\mathcal{H}^{n-1}$). But as we did in the case of the H_i's, the numbers $\bar{\eta}_{i,1,1}$ and $\bar{\eta}_{i,1,i+1}$ may be replaced by smaller numbers in the corresponding results, and thus, comparable weaker bounds can be obtained. We do not repeat all these arguments/results for ϱ_i since they are totally analogous to the ones for the H_i's, in order not to enlarge unnecessarily the paper. We just would like to show, as an example, the bound obtained for $\int_{\mathbb{S}^{n-1}} \varrho_i^2 \, \mathrm{d}\mathcal{H}^{n-1}$ taking $F(x) = 1/x$ in the second inequality of Theorem 4.1, as well as the corresponding weaker bound.

Corollary 4.1. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2. Then, for all $0 \le i \le n-1$,*

$$\int_{\mathbb{S}^{n-1}} \varrho_i^2 \, \mathrm{d}\mathcal{H}^{n-1} \ge \frac{n|B_2^n|\mathrm{W}_{n-i}^2}{|B_2^n|^2 - \mathrm{W}_{n-i}^2 \bar{\eta}_{i,1,i+1}^2} \ge \frac{n|B_2^n|\mathrm{W}_{n-i}^2}{|B_2^n|^2 - \dfrac{\left(\mathrm{W}_{n-1}\mathrm{W}_{n-i} - |B_2^n|\mathrm{W}_{n-i-1}\right)^2}{\mathrm{W}_{n-i}^2 \left(\bar{\mathrm{R}}(K-c_K) - \bar{\mathrm{r}}(K-c_K)\right)^2}}.$$

Equality holds, in the first inequality, for $K = B_2^n$ (up to dilations).

If we intend to provide a lower bound for the integral of $F \circ \varrho_i$ in terms of the quermaßintegrals only, an additional hypothesis has to be imposed. The following result is a direct consequence of Theorem 4.1.

Corollary 4.2. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2 and $0 \le i \le n-1$ be such that*

$$(4.2) \quad \max\left\{\bar{\mathrm{R}} - \frac{\mathrm{W}_{n-1}}{|B_2^n|}, \frac{\mathrm{W}_{n-1}}{|B_2^n|} - \bar{\mathrm{r}}\right\} \le \left(\frac{\mathrm{W}_{n-1}\mathrm{W}_{n-i}}{|B_2^n|^2} - \frac{\mathrm{W}_{n-i-1}}{|B_2^n|}\right)^{1/(i+1)},$$

and let

$$\eta = \frac{\mathrm{W}_{n-1}\mathrm{W}_{n-i}}{|B_2^n|^2} - \frac{\mathrm{W}_{n-i-1}}{|B_2^n|}.$$

Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, we have

$$\int_{\mathbb{S}^{n-1}} (F \circ \varrho_i) \, \mathrm{d}\mathcal{H}^{n-1} \ge n|B_2^n| \frac{F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} + \eta^{i/(i+1)}\right) + F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} - \eta^{i/(i+1)}\right)}{2}$$

Equality holds for $K = B_2^n$ (up to dilations).

We observe that the power $1/(i+1)$ in (4.2) is needed in order to keep the homogeneity in both sides of the inequality, because the $(n-i)$-th quermaßintegral is an homogeneous functional of degree i (see e.g. [8, Theorem 6.13]).

Remark 3. *If $K \in \mathcal{K}_0^2$ is a planar centered convex body of class C_+^2, then assumption (4.2) always holds for $i = 1$, as a consequence of the well-known Bonnesen inequality (see e.g. [16, p. 388]): indeed, (4.2) translates into*

$$\mathrm{p}^2 - 4\pi\mathrm{A} \ge 4\pi^2 \max\left\{\mathrm{R} - \frac{\mathrm{p}}{2\pi}, \frac{\mathrm{p}}{2\pi} - \mathrm{r}\right\}^2$$

when $n = 2$ and $i = 1$, which, depending on the value of the maximum, can be rewritten as either $\mathrm{A} - \mathrm{pR} + \pi\mathrm{R}^2 \le 0$ or $\mathrm{A} - \mathrm{pr} + \pi\mathrm{r}^2 \le 0$; both inequalities (known as Bonnesen's inequalities) always hold.

So, it is a natural question to ask wether this condition is also verified in higher dimensions, at least for the class of C_+^2 centered or 0-symmetric convex bodies. Next we answer this question in the negative, and thus, assumption (4.2) cannot be removed. To this end, we need an additional definition. A *cap-body* (of the ball) is the convex hull of B_2^n and countably many points such that the line segment joining any pair of those points intersects B_2^n. Cap-bodies are particular cases of the more general defined *p-tangential bodies* for which we refer to [16, s. 2.2]. It is well known (see [5], [16, Theorem 7.6.17]) that if $K \in \mathcal{K}_0^n$ with $B_2^n \subset K$, then K is a cap-body of B_2^n if and only if $|K| = \mathrm{W}_0 = \mathrm{W}_1 = \cdots = \mathrm{W}_{n-1}$; in particular, $\mathrm{r} = 1$.

Proposition 4.1. *There exist 0-symmetric convex bodies $K \in \mathcal{K}_0^n$, $n \geq 3$, of class C_+^2, such that, for all $0 \leq i \leq n-1$,*

$$(4.3) \quad \max\left\{\mathrm{R} - \frac{\mathrm{W}_{n-1}}{|B_2^n|}, \frac{\mathrm{W}_{n-1}}{|B_2^n|} - \mathrm{r}\right\} > \left(\frac{\mathrm{W}_{n-1}\mathrm{W}_{n-i}}{|B_2^n|^2} - \frac{\mathrm{W}_{n-i-1}}{|B_2^n|}\right)^{1/(i+1)}.$$

Proof. If $i = 0$ then any convex body different from a ball verifies the required condition, and so we assume throughout the proof that $1 \leq i \leq n-1$.

Let $K = \mathrm{conv}\{B_2^n, Re_1, -Re_1\}$, $R > 1$, be a 0-symmetric cap-body, for which $|K| = \mathrm{W}_0 = \cdots = \mathrm{W}_{n-1}$, $\mathrm{R} = R$ and $\mathrm{r} = 1$. We want to prove that there exists $R > 1$ such that, for all $1 \leq i \leq n-1$,

$$(4.4) \qquad \max\left\{R - \frac{|K|}{|B_2^n|}, \frac{|K|}{|B_2^n|} - 1\right\} > \left(\frac{|K|^2}{|B_2^n|^2} - \frac{|K|}{|B_2^n|}\right)^{1/(i+1)}.$$

To this end, we first observe that the condition

$$\frac{|K|}{|B_2^n|} - 1 > \left(\frac{|K|^2}{|B_2^n|^2} - \frac{|K|}{|B_2^n|}\right)^{1/(i+1)}$$

is equivalent to $\left(|K|/|B_2^n| - 1\right)^i > |K|/|B_2^n|$, and hence, if $i \geq 2$, then for $|K|$ large enough (indeed it suffices that $|K|/|B_2^n| \geq 3$), i.e., for R large enough, condition (4.4) holds.

So we assume $i = 1$, and we have to show that there exists $R > 1$ such that $R - |K|/|B_2^n| > \left(|K|^2/|B_2^n|^2 - |K|/|B_2^n|\right)^{1/2}$, or equivalently, such that

$$(4.5) \qquad\qquad |K| < |B_2^n|\frac{R^2}{2R-1}.$$

It can be seen (see e.g. [9]) that the volume of such a cap-body is given by

$$|K| = 2|B_2^{n-1}|\left(\frac{(R^2-1)^{(n+1)/2}}{nR^n} + \int_0^{1/R}(1-x^2)^{(n-1)/2}\mathrm{d}x\right),$$

and hence we have to show that there exists $R > 1$ such that

$$\frac{2R-1}{R^2}\left(\frac{(R^2-1)^{(n+1)/2}}{nR^n} + \int_0^{1/R}\left(1-x^2\right)^{(n-1)/2}\mathrm{d}x\right) < \frac{|B_2^n|}{2|B_2^{n-1}|}$$

for all $n \geq 3$. If $n = 3$ the above condition reduces to $R^2 + 1 > 2R$, which always holds. Moreover, $\int_0^{1/R}(1-x^2)^{(n-1)/2}\,\mathrm{d}x$ is decreasing in R, and hence

$$\int_0^{1/R}\left(1-x^2\right)^{(n-1)/2}\mathrm{d}x < \int_0^1\left(1-x^2\right)^{(n-1)/2}\mathrm{d}x = \frac{|B_2^n|}{2|B_2^{n-1}|}$$

(see (3.2)). Therefore, it suffices to see that

$$\frac{2R-1}{R^2}\left(\frac{(R^2-1)^{(n+1)/2}}{nR^n} + \frac{|B_2^n|}{2|B_2^{n-1}|}\right) < \frac{|B_2^n|}{2|B_2^{n-1}|}$$

for some $R > 1$, or equivalently,

$$\frac{(2R-1)(R+1)^{(n+1)/2}(R-1)^{(n-3)/2}}{R^n} < \frac{n}{2}\frac{|B_2^n|}{|B_2^{n-1}|},$$

which is trivially true, for all $n \geq 4$, if $R > 1$ is sufficiently close to 1. It proves (4.5) and hence (4.4).

Thus we have shown that there exists a 0-symmetric cap-body $K \in \mathcal{K}_0^n$ satisfying inequality (4.3). Since every convex body can be approximated by convex bodies of class C_+^2 (with respect to the Hausdorff metric, see [16, s. 3.4]), we can find, on the one hand, a sequence $(K_j)_{j\in\mathbb{N}} \subset \mathcal{K}_0^n$ of C_+^2 sets with $\lim_{j\to\infty} K_j = K$. Moreover, the sequence $\left(K_j^c = (K_j - K_j)/2\right)_{j\in\mathbb{N}}$, which is also formed by C_+^2 convex bodies, satisfies that

$$\lim_{j\to\infty}\frac{1}{2}(K_j - K_j) = \frac{1}{2}(K - K) = K,$$

because K is 0-symmetric. Hence, we have a sequence $(K_j^c)_{j\in\mathbb{N}}$ of 0-symmetric convex bodies of class C_+^2 with limit K. On the other hand, since the quermaßintegrals, circumradius and inradius are continuous functionals (with respect to the Hausdorff metric, see e.g. [8, Theorem 6.13]), then we can conclude that there exists a 0-symmetric convex body $K_j^c \in \mathcal{K}_0^n$ of class C_+^2, for j large enough, satisfying (4.3) for all $0 \leq i \leq n-1$. □

5. RADII OF CURVATURE AND ROOTS OF STEINER POLYNOMIALS

On the one hand, in [10], the roots of the Steiner polynomial of a convex body K (cf. (3.1)), considered as a formal polynomial in a complex variable $z \in \mathbb{C}$, are related to the maximum and minimum value of its principal radii of curvature. On the other hand, in [7], lower estimates for the integrals of convex functions of the curvature of a smooth bounded planar convex curve are shown in terms of the roots of the Steiner polynomial of the region enclosed by the curve.

Following this idea, in this section we are going to relate the integral of a transformation of the i-th normalized elementary symmetric function of the principal radii of curvature of a C_+^2 convex body K, to the roots of its the Steiner polynomial.

To this end, for $K \in \mathcal{K}_0^n$, we denote by $f_K(z) = \sum_{i=0}^n \binom{n}{i} W_i z^i$ the Steiner polynomial of K. Moreover, let $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$ be the roots of $f_K(z)$ and, for the sake of brevity, we write $s_i(\gamma) = s_i(\gamma_1, \ldots, \gamma_n)$.

Proposition 5.1. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2, with $c_K = 0$ and $K \neq B_2^n(r)$ for any $r > 0$. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $0 \leq i \leq n - 1$,*

$$\int_{\mathbb{S}^{n-1}} (F \circ \varrho_i)\, \mathrm{d}\mathcal{H}^{n-1} \geq \frac{n|B_2^n|}{2} \left[F\left(\left| s_i(\gamma) \right| + \frac{\left| s_i(\gamma) s_1(\gamma) \right| - \left| s_{i+1}(\gamma) \right|}{\bar{R} - \bar{r}} \right) \right.$$
$$\left. + F\left(\left| s_i(\gamma) \right| - \frac{\left| s_i(\gamma) s_1(\gamma) \right| - \left| s_{i+1}(\gamma) \right|}{\bar{R} - \bar{r}} \right) \right].$$

Proof. From $f_K(z) = |B_2^n| \prod_{i=1}^n (z - \gamma_i)$ we get

$$(5.1) \qquad \binom{n}{i} W_{n-i} = (-1)^i |B_2^n| \sigma_i(\gamma_1, \ldots, \gamma_n), \quad \text{i.e.,} \quad \frac{W_{n-i}}{|B_2^n|} = \left| s_i(\gamma) \right|.$$

Then, using Theorem 4.1, together with (3.4), we get the result. $\qquad \square$

Remark 4. *When $i = n - 1$, the argument of the function F in the above expression can be expressed in terms of the real parts of the roots and their inverses: denoting by $\mathrm{Re}\, z$ the real part of a complex number $z \in \mathbb{C}$, we get*

$$\left| s_{n-1}(\gamma) \right| + \frac{\left| s_{n-1}(\gamma) s_1(\gamma) \right| - \left| s_n(\gamma) \right|}{\bar{R} - \bar{r}}$$
$$= \left| s_n(\gamma) \right| \left[\left| \frac{s_{n-1}(\gamma)}{s_n(\gamma)} \right| + \frac{\left| \frac{s_{n-1}(\gamma)}{s_n(\gamma)} s_1(\gamma) \right| - 1}{\bar{R} - \bar{r}} \right]$$
$$= \left| s_n(\gamma) \right| \left[\left| s_1\left(\frac{1}{\gamma} \right) \right| + \frac{\left| s_1\left(\frac{1}{\gamma} \right) s_1(\gamma) \right| - 1}{\bar{R} - \bar{r}} \right]$$
$$= \left| \gamma_1 \cdots \gamma_n \right| \left[\left| \sum_{i=1}^n \mathrm{Re}\, \frac{1}{\gamma_i} \right| + \frac{\left| \sum_{i=1}^n \mathrm{Re}\, \frac{1}{\gamma_i} \right| \left| \sum_{i=1}^n \mathrm{Re}\, \gamma_i \right| - 1}{\bar{R} - \bar{r}} \right].$$

For particular families of sets, the bounds provided in Sections 3 and 4 can be expressed in a more convenient way. We show here a couple of examples.

Proposition 5.2. *Let $K \in \mathcal{K}_0^n$ be of class C_+^2 such that $|K| = W_1 = W_2$. Then, for any $\alpha \geq 0$,*

$$\int_{\mathrm{bd}\, K} H^{\alpha+1} \mathrm{d}\mathcal{H}^{n-1} \geq n|K| \quad \text{and} \quad \int_{\mathrm{bd}\, K} \frac{1}{H^\alpha}\, \mathrm{d}\mathcal{H}^{n-1} \geq n|K|.$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

Similar results can be obtained for all H_i, $i = 2, \ldots, n - 1$.

On a rather different tack, a convex body $K \in \mathcal{K}_0^n$ is said to have *constant width* if it has the same width $\omega_K(u) = h_K(u) + h_K(-u)$ in all directions $u \in \mathbb{S}^{n-1}$. Constant width sets have been intensively studied along the last century. In the plane they are well known, whereas the situation becomes much more complicated in dimension $n \geq 3$.

It is well known (see e.g. [2, p. 68]) that if $K \in \mathcal{K}_0^n$ has constant width b, then the inball and the circumball of K are concentric and $R + r = b$. Thus, assuming that the origin is the incenter of K, we have $\bar{R} = R$ and $\bar{r} = r$. Moreover,

$$(5.2) \qquad \mathrm{b}\left(1 - \sqrt{\frac{n}{2(n+1)}}\right) \leq \mathrm{r} \leq \mathrm{R} \leq \mathrm{b}\sqrt{\frac{n}{2(n+1)}}$$

(see e.g. [2, p. 68, (7.3)]). For a nice and thorough survey on convex bodies of constant width see [2].

Using Theorem 4.1, the following type of results can be obtained for constant width sets. For the sake of brevity we write $c_n = \sqrt{2n/(n+1)} - 1$.

Theorem 5.1. *Let $K \in \mathcal{K}_0^n$ be a C_+^2 constant width set of width b. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $0 \leq i \leq n - 1$,*

$$\int_{\mathbb{S}^{n-1}} (F \circ \varrho_i) \, \mathrm{d}\mathcal{H}^{n-1} \geq \frac{n|B_2^n|}{2} \left[F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} + \frac{\mathrm{b}\mathrm{W}_{n-i} - 2\mathrm{W}_{n-i-1}}{c_n|B_2^n|\,\mathrm{b}}\right) \right.$$
$$\left. + F\left(\frac{\mathrm{W}_{n-i}}{|B_2^n|} - \frac{\mathrm{b}\mathrm{W}_{n-i} - 2\mathrm{W}_{n-i-1}}{c_n|B_2^n|\,\mathrm{b}}\right) \right].$$

Moreover,

$$\int_{\mathbb{S}^{n-1}} (F \circ \varrho_i) \, \mathrm{d}\mathcal{H}^{n-1} \geq \frac{n|B_2^n|}{2} \left[F\left(\frac{c_n + 1}{c_n}\left|\mathrm{s}_i(\gamma)\right| - \frac{2\left|\mathrm{s}_{i+1}(\gamma)\right|}{c_n\mathrm{b}}\right) \right.$$
$$\left. + F\left(\frac{c_n - 1}{c_n}\left|\mathrm{s}_i(\gamma)\right| + \frac{2\left|\mathrm{s}_{i+1}(\gamma)\right|}{c_n\mathrm{b}}\right) \right].$$

Equality holds in both inequalities if $K = B_2^n$.

Proof. Since $\mathrm{W}_{n-1}/|B_2^n| = \mathrm{b}/2$ and $R + r = \mathrm{b}$, using (5.2) we get

$$\frac{\mathrm{W}_{n-1}}{|B_2^n|} - \mathrm{r} = \frac{\mathrm{b}}{2} - \mathrm{r} = \frac{\mathrm{b}}{2} - (\mathrm{b} - \mathrm{R}) = \mathrm{R} - \frac{\mathrm{b}}{2} \leq \frac{\mathrm{b}}{2}\left(\sqrt{\frac{2n}{n+1}} - 1\right),$$

and therefore

$$\bar{\eta}_{i,1,1} = \frac{\mathrm{W}_{n-i}\mathrm{W}_{n-1} - |B_2^n|\mathrm{W}_{n-i-1}}{|B_2^n|^2 \max\left\{R - \frac{\mathrm{W}_{n-1}}{|B_2^n|}, \frac{\mathrm{W}_{n-1}}{|B_2^n|} - \mathrm{r}\right\}} = \frac{\mathrm{b}\mathrm{W}_{n-i} - 2\mathrm{W}_{n-i-1}}{2|B_2^n|\left(R - \frac{\mathrm{b}}{2}\right)}$$
$$\geq \frac{\mathrm{b}\mathrm{W}_{n-i} - 2\mathrm{W}_{n-i-1}}{c_n|B_2^n|\,\mathrm{b}}.$$

Using the above bound, the first inequality is a direct consequence of Theorem 4.1; the second inequality follows from the first one and (5.1). $\square$

6. Radial function and dual quermassintegrals

A non-empty set $S \subset \mathbb{R}^n$ is called *starshaped* (with respect to 0) if the line segment $[0, x] \subseteq S$ for all $x \in S$. For a compact starshaped set K, the radial function is defined as

$$\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in \mathbb{R}^n \backslash \{0\},$$

and clearly, $\rho_K(u)u \in \operatorname{bd} K$. We will denote by $\mathcal{S}_0^n$ the family of all compact starshaped sets in $\mathbb{R}^n$ having the origin as an interior point.

Dual quermaßintegrals (and dual mixed volumes) were introduced by Lutwak in [12], being the starting point for the development of the nowadays known as dual Brunn-Minkowski theory (see e.g. [16, s. 9.3]). The dual quermaßintegral of order i, $i = 0, \dots, n$, of $K \in \mathcal{S}_0^n$ is defined as

$$(6.1) \qquad \widetilde{W}_i(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^{n-i} \, d\mathcal{H}^{n-1},$$

which is monotonous and homogeneous of degree $n - i$ (see e.g. [6, s. A.7]), although not translation invariant. In particular, the use of spherical coordinates immediately yields $\widetilde{W}_0(K) = |K|$, whereas $\widetilde{W}_n(K) = |B_2^n|$ and $2\widetilde{W}_{n-1}(K)/|B_2^n|$ is the average length of chords of K through the origin. Moreover, if $K \in \mathcal{K}_0^n$, then $\widetilde{W}_i(K) \leq W_i(K)$ for all $i = 0, \dots, n$ (see [12]).

Let $K \in \mathcal{S}_0^n$. Again, for the sake of brevity, we will write $\widetilde{W}_i = \widetilde{W}_i(K)$, and for any $0 \leq j, k, l \leq n$ with $j + k \leq n$, let

$$\widetilde{\eta}_{j,k,l} = \frac{|K||B_2^n| - \widetilde{W}_{n-j}\widetilde{W}_j}{\widetilde{W}_{n-l}^2 \max\left\{\bar{R}^{n-j} - \frac{\widetilde{W}_k}{\widetilde{W}_{n-j+k}}, \frac{\widetilde{W}_k}{\widetilde{W}_{n-j+k}} - \bar{r}^{n-j}\right\}} \quad \text{if } K \neq B_2^n(r), \ r > 0,$$

$$\widetilde{\eta}_{j,k,l} = 0 \qquad \text{if } K = B_2^n(r) \text{ for some } r > 0,$$

where $\bar{R}, \bar{r}$ are defined analogously to the convex case. From the monotonicity of the dual quermassintegrals we get $\bar{r}^{n-j} \widetilde{W}_{n-j+k} \leq \widetilde{W}_k \leq \bar{R}^{n-j} \widetilde{W}_{n-j+k}$ (cf. (3.4)), and so the above maximum is always positive. This, together with the relation $|K||B_2^n| \geq \widetilde{W}_{n-j}\widetilde{W}_j$ (a consequence of the dual Aleksandrov-Fenchel inequalities, see e.g. [12, Theorem 2]), ensures that $\widetilde{\eta}_{j,k,l} \geq 0$.

Theorem 6.1. *Let $K \in \mathcal{S}_0^n$. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $0 \leq i \leq n$,*

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \rho_K^i\right) d\mathcal{H}^{n-1} \geq n|B_2^n| \frac{F\left(\frac{\widetilde{W}_{n-i}}{|B_2^n|} + \widetilde{\eta}_{i,i,0}\right) + F\left(\frac{\widetilde{W}_{n-i}}{|B_2^n|} - \widetilde{\eta}_{i,i,0}\right)}{2},$$

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \frac{1}{\rho_K^i}\right) \rho_K^i \, d\mathcal{H}^{n-1} \geq n\widetilde{W}_{n-i} \frac{F\left(\frac{|B_2^n|}{\widetilde{W}_{n-i}} + \widetilde{\eta}_{i,0,i}\right) + F\left(\frac{|B_2^n|}{\widetilde{W}_{n-i}} - \widetilde{\eta}_{i,0,i}\right)}{2}.$$

Equality holds in both inequalities if $K = B_2^n$ (up to dilations).

Proof. In order to prove (6.1) we apply Proposition 1.1 to the probability space $\left(\mathbb{S}^{n-1}, \mathcal{H}^{n-1}/(n|B_2^n|)\right)$ and the functions $\rho = \rho_K^i$ and $h = \rho_K^{n-i}$. Then, by (6.1), we get $\mathbb{E}\rho = \widetilde{W}_{n-i}/|B_2^n|$, $\mathbb{E}h = \widetilde{W}_i/|B_2^n|$ and

$$\operatorname{Cov}(\rho, h) = \frac{|K||B_2^n| - \widetilde{W}_{n-i}\widetilde{W}_i}{|B_2^n|^2}.$$

Moreover, since $\rho_K(u)u \in \operatorname{bd} K$, we have

$$\|h - \mathbb{E}h\|_\infty = \sup \left\{ \left| \rho_K(u)^{n-i} - \frac{\widetilde{W}_i}{|B_2^n|} \right| : u \in \mathbb{S}^{n-1} \right\}$$

$$= \max \left\{ \bar{R}^{n-i} - \frac{\widetilde{W}_i}{|B_2^n|}, \frac{\widetilde{W}_i}{|B_2^n|} - \bar{r}^{n-i} \right\}.$$

Altogether shows the first inequality.

Second inequality is obtained analogously, but now as a consequence of Proposition 2.1 for $\rho = 1/\rho_K^i$, $h = \rho_K^{n-i}$ and $g = \rho_K^i/(n\widetilde{W}_{n-i})$. The equality case is trivial. $\qquad\square$

Remark 5. *We observe that, since (6.1) can be defined for any $i \in \mathbb{R}$, Theorem 6.1 holds true for all $i \in \mathbb{R}$, just properly defining the values $\widetilde{\eta}_{i,j,k}$. Moreover, taking $F(x) = x^{\alpha+1}$ or $F(x) = 1/x^\alpha$ for suitable powers $\alpha \geq 0$, and considering (6.1) defined for any $i \in \mathbb{R}$, new inequalities relating the dual quermaßintegrals with the in- and outer radii can be obtained.*

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